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Calculation of the relativistic bulk tensor and shear tensor of relativistic accretion flows in the Kerr metric.

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Abstract. In this paper, we calculate the relativistic bulk tensor and shear tensor of the relativistic accretion flows in the Kerr metric, overall and without any approximation. We obtain the relations of all components of the relativistic bulk and shear tensor in terms of components of four-velocity and its derivatives, Christoffel symbols and metric components in the BLF. Then, these components are derived in the equatorial plane. To see the behavior of the relativistic bulk tensor and shear tensor in the relativistic accretion disks around the rotating black holes, we introduce a radial form for the radial component of the four-velocity in the LNRF; therefore, all components of the bulk and shear tensor are derived in the BLF. Figures of non-zero components of the bulk tensor and shear tensor are shown for some states. We use the radial model to study the importance and influence of the relativistic bulk tensor in the accretion disks around the rotating black holes. Also, we see that in some cases bulk tensor may be important and comparable with the shear tensor. Especially, we see that bulk tensor in the inner radii is more important.

Keywords: relativistic viscosity, relativistic bulk tensor, relativistic shear tensor, relativistic shear stress viscosity, black holes accretion disks, relativistic flows.

1 Introduction

The accretion disks around the rotating black holes are studied by some scholars. Because of strong gravity of black holes, the relativistic method must be used. An important parameter for the energy distribution is shear stress viscosity. Abramowicz et al. [1] used a standard and nonrelativistic form for viscosity to study ADAFs around the Kerr metric. Peitz & Appl [11] calculated the $r\phi$ component of the shear tensor to study the transonic viscous accretion disks in the Kerr metric. Gammie & Popham [5] studied the thin ADAF disks around the Kerr black holes and they calculated the shear tensor by using the non-relativistic and relativistic casual viscosity. Takahashi [12] study the stationary hydrodynamic equations of the three states of the accretion disks around the Kerr black holes with calculating non-relativistic and relativistic causal viscosity. Moeen et al. [9] calculated two components of the shear tensor by introducing a zero radial component for the four-velocity; then, they solved the equations of the optically thin, steady state, accretion disks around the Kerr black holes analytically. Moeen [8] calculated the bulk and shear tensor for accretion disks around the non-rotating black holes with introducing a simple model for radial velocity and shows that bulk viscosity in those disks may be important.

It is not clear that how the shear stress viscosity affects the relativistic accretion flows. Shear stress viscosity includes two types of viscosity which are bulk and shear viscosity. In Mihalas and Mihalas [6] the relations of all non-relativistic components of shear and bulk tensor with velocity are calculated. Also, in the non-relativistic study, Narayan and Yi [10] have used these components to consider the axisymmetric, steady-state accretion flow around the central mass. In this paper, we derive the general relations of relativistic shear stress viscosity of the stationary, axisymmetric, accretion flows in the Kerr metric. So, we try to derive some explicit and useful form for the components of bulk and shear tensor.

Metric, units and reference frames are explained in section 2. The relativistic bulk tensor is derived in section 3. All components of the relativistic shear tensor are acquired in section 4. The components of shear and bulk tensor in the equatorial plane are calculated in section 5. In section 6, we introduce a radial model for the radial component of the four-velocity in the LNRF. Then, the other components of the four-velocity in the BLF are derived. Also, the time dilation is used to checking the radial model. Components of the shear and bulk tensors for different values of parameter n are calculated in section 7 and figures of those components are in this section. Influence of the bulk tensor, summary and conclusion are seen in section 8.

2 Metric, units, and reference frames

2.1 Background metric and units

For background geometry around the rotating black hole, we use the Kerr metric in the Boyer-Lindquist coordinate.

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{1}$$

where $\alpha, \beta = r, \theta$ and ϕ. We use $m = GM/c^2$, $\Sigma = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$ and $A = \Sigma\Delta + 2mr(r^2 + a^2)$, where G is the gravitational constant, M is the black hole mass, c is the speed of light and J is the angular momentum of the black hole ($a = Jc/GM^2$, $-1 < a < 1$). Therefore, the components of metric, $g_{\alpha\beta}$, and its inverse, $g^{\alpha\beta}$ in the Boyer-Lindquist coordinate are given as [9]

$$\begin{aligned} g_{tt} &= -(1 - \frac{2mr}{\Sigma}), \qquad g_{rr} = \frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = \Sigma, \\ g_{\phi\phi} &= \frac{Asin^2\theta}{\Sigma}, \qquad g_{t\phi} = g_{\phi t} = -\frac{2marsin^2\theta}{\Sigma}, \end{aligned} \tag{2}$$

$$\begin{aligned} g^{tt} &= -(\frac{A}{\Delta\Sigma}), \qquad g^{rr} = \frac{\Delta}{\Sigma}, \qquad g^{\theta\theta} = \frac{1}{\Sigma}, \\ g^{\phi\phi} &= \frac{1}{\Delta sin^2\theta}(1 - \frac{2mr}{\Sigma}), \qquad g^{t\phi} = g^{\phi t} = -\frac{2mar}{\Sigma\Delta}. \end{aligned} \tag{3}$$

For units similar to [5], we use $G = M = c = 1$. This scaling will be used in all calculation, so all figures will be shown versus dimensionless parameter r/m

2.2 Reference frames

In this paper, two reference frames are used; the Boyer-Lindquist reference frame and the locally non-rotating reference frame. In the most calculation, we use the Boyer-Lindquist reference frame (BLF) based on the Boyer-Lindquist coordinates of the Kerr metric. The locally non-rotating reference frame (LNRF) which is formed by observers with a future-directed unit vector orthogonal to $t = constant$. In the Boyer-Lindquist coordinate, the locally non-rotating reference frame LNRF observer is moving with the angular velocity of frame dragging ($\omega = -g_{t\phi}/g_{\phi\phi} = \frac{2mar}{A}$). We use the hat for the physical quantities measured in the LNRF such as $u^{\hat{\mu}}$ and $u_{\hat{\mu}}$. The transformation matrixes between the LNRF and the BLF are shown in Appendix A.

3 Relativistic bulk tensor

In the relativistic Navier-Stokes flow, the relativistic bulk viscosity is written as [7]:

$$B^{\mu\nu} = -\zeta b^{\mu\nu}, \tag{4}$$

where ζ is the coefficient of the bulk viscosity and the bulk tensor $b^{\mu\nu}$ is as follows

$$b^{\mu\nu} = \Theta h^{\mu\nu}. \tag{5}$$

The projection tensor ($h^{\mu\nu}$) and the expansion of the fluid world line (Θ) can be obtained from the following form [8]

$$\begin{aligned} h^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu, \\ \Theta &= u^\gamma_{;\gamma} = \frac{\partial u^\gamma}{\partial x^\gamma} + \Gamma^\nu_{\gamma\nu} u^\gamma, \end{aligned} \tag{6}$$

where u^μ is the four velocity and u^μ is the covariant four velocity. $h^{\mu\nu}$ is the symmetric tensor because the metric is symmetric, i.e. $g^{\mu\nu} = g^{\nu\mu}$; also, Θ is symmetric (it is a scalar or a tensor of rank zero), so the bulk tensor $b^{\mu\nu}$ is symmetric.

Since we study the stationary axisymmetric accretion flow, $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ are non-zero partial derivatives. So, the expansion of fluid world line in the Kerr metric can be obtained from following form

$$\Theta = \frac{\partial u^\gamma}{\partial x^\gamma} + u^r(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}). \tag{7}$$

So we have

$$\begin{aligned} \Theta &= \frac{\partial u^t}{\partial t} + \frac{\partial u^r}{\partial r} + \frac{\partial u^\theta}{\partial \theta} + \frac{\partial u^\phi}{\partial \phi} + u^r(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) \\ &= \frac{\partial u^r}{\partial r} + \frac{\partial u^\theta}{\partial \theta} + u^r(\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}). \end{aligned} \tag{8}$$

The radial component of the four-velocity which we call radial velocity and the azimuthal component of four velocity are important components in the bulk tensor. So, we can calculate all components of bulk tensor in the Kerr metric as

$$\begin{aligned} &b^{tt} = \Theta h^{tt} = \Theta(g^{tt} + (u^t)^2), \quad b^{rr} = \Theta h^{rr} = \Theta(g^{rr} + (u^r)^2), \quad b^{\theta\theta} = \Theta h^{\theta\theta} = \Theta(g^{\theta\theta} + (u^\theta)^2), \\ &b^{\phi\phi} = \Theta h^{\phi\phi} = \Theta(g^{\phi\phi} + (u^\phi)^2), \quad b^{tr} = b^{rt} = \Theta h^{rt} = \Theta u^r u^t, \quad b^{t\theta} = b^{\theta t} = \Theta h^{t\theta} = \Theta u^t u^\theta, \\ &b^{t\phi} = b^{\phi t} = \Theta h^{t\phi} = \Theta(g^{t\phi} + u^t u^\phi), \quad b^{r\theta} = b^{\theta r} = \Theta h^{r\theta} = \Theta u^r u^\theta, \quad b^{r\phi} = b^{\phi r} = \Theta h^{r\phi} = \Theta u^r u^\phi, \\ &b^{\theta\phi} = b^{\phi\theta} = \Theta h^{\theta\phi} = \Theta u^\theta u^\phi. \end{aligned} \tag{9}$$

4 Relativistic shear tensor

In the relativistic Navier-Stokes flow, the relativistic shear viscosity ($S^{\mu\nu}$) is given as follows [7]

$$S^{\mu\nu} = -2\lambda\sigma^{\mu\nu}, \tag{10}$$

where λ is the coefficient of the dynamic viscosity; also, the shear tensor ($\sigma^{\mu\nu}$) of the fluid is written as

$$\sigma^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \sigma_{\alpha\beta} \quad . \tag{11}$$

In this relation, the shear rate, $\sigma_{\alpha\beta}$ is introduced as [12]

$$\sigma_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\gamma} h^{\gamma}_{\beta} + u_{\beta;\gamma} h^{\gamma}_{\alpha}) - \frac{1}{3}\Theta h_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} + u_{\nu;\mu} + a_{\mu} u_{\nu} + a_{\nu} u_{\mu}) - \frac{1}{3}\Theta h_{\mu\nu}, \tag{12}$$

where $a_{\mu} = u_{\mu;\gamma} u^{\gamma}$ is the four acceleration. The shear rate $\sigma_{\alpha\beta}$ is symmetric because the expressions $u_{\mu;\nu} + u_{\nu;\mu} + a_{\mu} u_{\nu} + a_{\nu} u_{\mu}$ and $h_{\mu\nu}$ are symmetric. Therefore, the form of the shear tensor will be as follows

$$\begin{aligned}\sigma^{\mu\nu} &= g^{\mu\alpha} g^{\nu\beta} \left(\frac{1}{2}(u_{\alpha;\gamma} h^{\gamma}_{\beta} + u_{\beta;\gamma} h^{\gamma}_{\alpha}) - \frac{1}{3}\Theta h_{\mu\nu}\right) = \frac{1}{2}(u^{\mu}_{;\gamma} h^{\gamma\nu} + u^{\nu}_{;\gamma} h^{\gamma\mu}) - \frac{1}{3}\Theta h^{\mu\nu} \\ &= \frac{1}{2}[(u^{\mu}_{,\gamma} + \Gamma^{\mu}_{\gamma\lambda} u^{\lambda}) h^{\gamma\nu} + (u^{\nu}_{,\gamma} + \Gamma^{\nu}_{\gamma\lambda} u^{\lambda}) h^{\gamma\mu}] - \frac{1}{3}\Theta h^{\mu\nu}. \end{aligned} \tag{13}$$

All components of the relativistic shear tensor are calculated as

$$\begin{aligned}
\sigma^{tt} &= (u^t_{,r} + \Gamma^t_{rt} u^t + \Gamma^t_{rr} u^r + \Gamma^t_{r\phi} u^{\phi}) h^{rt} + (\Gamma^t_{tt} u^t + \Gamma^t_{tr} u^r + \Gamma^t_{t\phi} u^{\phi}) h^{tt} + (u^t_{,\theta} + \Gamma^t_{\theta t} u^t + \Gamma^t_{\theta r} u^r \\
&\quad + \Gamma^t_{\theta\phi} u^{\phi}) h^{t\theta} + (\Gamma^t_{\phi t} u^t + \Gamma^t_{\phi r} u^r + \Gamma^t_{\phi\phi} u^{\phi}) h^{t\phi} - \frac{1}{3}\Theta h^{tt}, \\
\sigma^{tr} &= \sigma^{rt} = \frac{1}{2}[(u^t_{,r} + \Gamma^t_{rt} u^t + \Gamma^t_{rr} u^r + \Gamma^t_{r\phi} u^{\phi}) h^{rr} + (\Gamma^t_{tt} u^t + \Gamma^t_{tr} u^r + \Gamma^t_{t\phi} u^{\phi}) h^{rt} + (u^t_{,\theta} + \Gamma^t_{\theta t} u^t \\
&\quad + \Gamma^t_{\theta r} u^r + \Gamma^t_{\theta\phi} u^{\phi}) h^{\theta r} + (\Gamma^t_{\phi t} u^t + \Gamma^t_{\phi r} u^r + \Gamma^t_{\phi\phi} u^{\phi}) h^{r\phi} + (u^r_{,r} + \Gamma^r_{rt} u^t + \Gamma^r_{rr} u^r + \Gamma^r_{r\phi} u^{\phi}) h^{rt} \\
&\quad + (\Gamma^r_{tt} u^t + \Gamma^r_{tr} u^r + \Gamma^r_{t\phi} u^{\phi}) h^{tt} + (u^r_{,\theta} + \Gamma^r_{t\theta} u^t + \Gamma^r_{r\theta} u^r + \Gamma^r_{\phi\theta} u^{\phi}) h^{\theta t} + (\Gamma^r_{\phi t} u^t + \Gamma^r_{\phi r} u^r \\
&\quad + \Gamma^r_{\phi\phi} u^{\phi}) h^{t\phi}] - \frac{1}{3}\Theta h^{rt}, \\
\sigma^{t\theta} &= \sigma^{\theta t} = \frac{1}{2}[(u^t_{,r} + \Gamma^t_{rt} u^t + \Gamma^t_{rr} u^r + \Gamma^t_{r\phi} u^{\phi}) h^{r\theta} + (\Gamma^t_{tt} u^t + \Gamma^t_{tr} u^r + \Gamma^t_{t\phi} u^{\phi}) h^{t\theta} + (u^t_{,\theta} + \Gamma^t_{\theta t} u^t \\
&\quad + \Gamma^t_{\theta r} u^r + \Gamma^t_{\theta\phi} u^{\phi}) h^{\theta\theta} + (\Gamma^t_{\phi t} u^t + \Gamma^t_{\phi r} u^r + \Gamma^t_{\phi\phi} u^{\phi}) h^{\theta\phi} + (u^{\theta}_{,r} + \Gamma^{\theta}_{rt} u^t + \Gamma^{\theta}_{rr} u^r + \Gamma^{\theta}_{r\phi} u^{\phi}) h^{rt} \\
&\quad + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta t} u^t + \Gamma^{\theta}_{\theta r} u^r + \Gamma^{\theta}_{\theta\phi} u^{\phi}) h^{\theta t} + (\Gamma^{\theta}_{tt} u^t + \Gamma^{\theta}_{tr} u^r + \Gamma^{\theta}_{t\phi} u^{\phi}) h^{tt} + (\Gamma^{\theta}_{\phi t} u^t + \Gamma^{\theta}_{\phi r} u^r \\
&\quad + \Gamma^{\theta}_{\phi\phi} u^{\phi}) h^{t\phi}] - \frac{1}{3}\Theta h^{t\theta}, \\
\sigma^{t\phi} &= \sigma^{\phi t} = \frac{1}{2}[(u^t_{,r} + \Gamma^t_{rt} u^t + \Gamma^t_{rr} u^r + \Gamma^t_{r\phi} u^{\phi}) h^{r\phi} + (\Gamma^t_{tt} u^t + \Gamma^t_{tr} u^r + \Gamma^t_{t\phi} u^{\phi}) h^{t\phi} + (u^t_{,\theta} + \Gamma^t_{\theta t} u^t \\
&\quad + \Gamma^t_{\theta r} u^r + \Gamma^t_{\theta\phi} u^{\phi}) h^{\theta\phi} + (\Gamma^t_{\phi t} u^t + \Gamma^t_{\phi r} u^r + \Gamma^t_{\phi\phi} u^{\phi}) h^{\phi\phi} + (u^{\phi}_{,r} + \Gamma^{\phi}_{rt} u^t + \Gamma^{\phi}_{rr} u^r + \Gamma^{\phi}_{r\phi} u^{\phi}) h^{rt} \\
&\quad + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t} u^t + \Gamma^{\phi}_{\theta r} u^r + \Gamma^{\phi}_{\theta\phi} u^{\phi}) h^{\theta t} + (\Gamma^{\phi}_{tt} u^t + \Gamma^{\phi}_{tr} u^r + \Gamma^{\phi}_{t\phi} u^{\phi}) h^{tt} + (\Gamma^{\phi}_{\phi t} u^t + \Gamma^{\phi}_{\phi r} u^r \\
&\quad + \Gamma^{\phi}_{\phi\phi} u^{\phi}) h^{t\phi}] - \frac{1}{3}\Theta h^{t\phi}, \\
\sigma^{rr} &= (u^r_{,r} + \Gamma^r_{rt} u^t + \Gamma^r_{rr} u^r + \Gamma^r_{r\phi} u^{\phi}) h^{rr} + (\Gamma^r_{tt} u^t + \Gamma^r_{tr} u^r + \Gamma^r_{t\phi} u^{\phi}) h^{rt} + (u^r_{,\theta} + \Gamma^r_{\theta t} u^t + \Gamma^r_{\theta r} u^r \\
&\quad + \Gamma^r_{\theta\phi} u^{\phi}) h^{\theta r} + (\Gamma^r_{\phi t} u^t + \Gamma^r_{\phi r} u^r + \Gamma^r_{\phi\phi} u^{\phi}) h^{r\phi} - \frac{1}{3}\Theta h^{rr}, \\
\sigma^{r\theta} &= \sigma^{\theta r} = \frac{1}{2}[(u^r_{,r} + \Gamma^r_{rt} u^t + \Gamma^r_{rr} u^r + \Gamma^r_{r\phi} u^{\phi}) h^{r\theta} + (\Gamma^r_{tt} u^t + \Gamma^r_{tr} u^r + \Gamma^r_{t\phi} u^{\phi}) h^{t\theta} + (u^r_{,\theta} + \Gamma^r_{\theta t} u^t \\
&\quad + \Gamma^r_{\theta r} u^r + \Gamma^r_{\theta\phi} u^{\phi}) h^{\theta\theta} + (\Gamma^r_{\phi t} u^t + \Gamma^r_{\phi r} u^r + \Gamma^r_{\phi\phi} u^{\phi}) h^{\theta\phi} + (u^{\theta}_{,r} + \Gamma^{\theta}_{rt} u^t + \Gamma^{\theta}_{rr} u^r + \Gamma^{\theta}_{r\phi} u^{\phi}) h^{rr} \\
&\quad + (\Gamma^{\theta}_{tt} u^t + \Gamma^{\theta}_{tr} u^r + \Gamma^{\theta}_{t\phi} u^{\phi}) h^{rt} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta t} u^t + \Gamma^{\theta}_{\theta r} u^r + \Gamma^{\theta}_{\theta\phi} u^{\phi}) h^{\theta r} + (\Gamma^{\theta}_{\phi t} u^t + \Gamma^{\theta}_{\phi r} u^r
\end{aligned}$$

$$+\Gamma^{\theta}_{\phi\phi}u^{\phi})h^{r\phi}] - \frac{1}{3}\Theta h^{r\theta},$$

$$\sigma^{r\phi} = \sigma^{\phi r} = \frac{1}{2}[(u^{r}_{,r} + \Gamma^{r}_{rt}u^{t} + \Gamma^{r}_{rr}u^{r} + \Gamma^{r}_{r\phi}u^{\phi})h^{r\phi} + (\Gamma^{r}_{tt}u^{t} + \Gamma^{r}_{tr}u^{r} + \Gamma^{r}_{t\phi}u^{\phi})h^{t\phi} + (u^{r}_{,\theta} + \Gamma^{r}_{\theta t}u^{t}$$
$$+\Gamma^{r}_{\theta r}u^{r} + \Gamma^{r}_{\theta\phi}u^{\phi})h^{\theta\phi} + (\Gamma^{r}_{\phi t}u^{t} + \Gamma^{r}_{\phi r}u^{r} + \Gamma^{r}_{\phi\phi}u^{\phi})h^{\phi\phi} + (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^{t} + \Gamma^{\phi}_{rr}u^{r} + \Gamma^{\phi}_{r\phi}u^{\phi})h^{rr}$$
$$+(\Gamma^{\phi}_{tt}u^{t} + \Gamma^{\phi}_{tr}u^{r} + \Gamma^{\phi}_{t\phi}u^{\phi})h^{rt} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^{t} + \Gamma^{\phi}_{\theta r}u^{r} + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta r} + (\Gamma^{\phi}_{\phi t}u^{t} + \Gamma^{\phi}_{\phi r}u^{r}$$
$$+\Gamma^{\phi}_{\phi\phi}u^{\phi})h^{r\phi}] - \frac{1}{3}\Theta h^{r\phi},$$

$$\sigma^{\theta\theta} = (u^{\theta}_{,r} + \Gamma^{\theta}_{rt}u^{t} + \Gamma^{\theta}_{rr}u^{r} + \Gamma^{\theta}_{r\phi}u^{\phi})h^{r\theta} + (\Gamma^{\theta}_{tt}u^{t} + \Gamma^{\theta}_{tr}u^{r} + \Gamma^{\theta}_{t\phi}u^{\phi})h^{t\theta} + (\Gamma^{\theta}_{\phi t}u^{t} + \Gamma^{\theta}_{\phi r}u^{r}$$
$$+\Gamma^{\theta}_{\phi\phi}u^{\phi})h^{\theta\phi} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta t}u^{t} + \Gamma^{\theta}_{\theta r}u^{r} + \Gamma^{\theta}_{\theta\phi}u^{\phi})h^{\theta\theta} - \frac{1}{3}\Theta h^{\theta\theta},$$

$$\sigma^{\theta\phi} = \sigma^{\phi\theta} = \frac{1}{2}[(u^{\theta}_{,r} + \Gamma^{\theta}_{rt}u^{t} + \Gamma^{\theta}_{rr}u^{r} + \Gamma^{\theta}_{r\phi}u^{\phi})h^{r\phi} + (\Gamma^{\theta}_{tt}u^{t} + \Gamma^{\theta}_{tr}u^{r} + \Gamma^{\theta}_{t\phi}u^{\phi})h^{t\phi} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta t}u^{t}$$
$$+\Gamma^{\theta}_{\theta r}u^{r} + \Gamma^{\theta}_{\theta\phi}u^{\phi})h^{\theta\phi} + (\Gamma^{\theta}_{\phi t}u^{t} + \Gamma^{\theta}_{\phi r}u^{r} + \Gamma^{\theta}_{\phi\phi}u^{\phi})h^{\phi\phi} + (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^{t} + \Gamma^{\phi}_{rr}u^{r} + \Gamma^{\phi}_{r\phi}u^{\phi})h^{\theta r}$$
$$+(\Gamma^{\phi}_{tt}u^{t} + \Gamma^{\phi}_{tr}u^{r} + \Gamma^{\phi}_{t\phi}u^{\phi})h^{\theta t} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^{t} + \Gamma^{\phi}_{\theta r}u^{r} + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta\theta} + (\Gamma^{\phi}_{\phi t}u^{t} + \Gamma^{\phi}_{\phi r}u^{r}$$
$$+\Gamma^{\phi}_{\phi\phi}u^{\phi})h^{\theta\phi}] - \frac{1}{3}\Theta h^{\theta\phi},$$

$$\sigma^{\phi\phi} = (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^{t} + \Gamma^{\phi}_{rr}u^{r} + \Gamma^{\phi}_{r\phi}u^{\phi})h^{r\phi} + (\Gamma^{\phi}_{tt}u^{t} + \Gamma^{\phi}_{tr}u^{r} + \Gamma^{\phi}_{t\phi}u^{\phi})h^{t\phi} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^{t} + \Gamma^{\phi}_{\theta r}u^{r}$$
$$+\Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta\phi} + (\Gamma^{\phi}_{\phi t}u^{t} + \Gamma^{\phi}_{\phi r}u^{r} + \Gamma^{\phi}_{\phi\phi}u^{\phi})h^{\phi\phi} - \frac{1}{3}\Theta h^{\phi\phi}. \tag{14}$$

By using the Christoffel symbols of Appendix B, we obtain

$$\sigma^{tt} = (u^{t}_{,r} + \Gamma^{t}_{rt}u^{t} + \Gamma^{t}_{r\phi}u^{\phi})h^{rt} + \Gamma^{t}_{tr}u^{r}h^{tt} + (u^{t}_{,\theta} + \Gamma^{t}_{\theta t}u^{t} + \Gamma^{t}_{\theta\phi}u^{\phi})h^{t\theta} + \Gamma^{t}_{\phi r}u^{r}h^{t\phi} - \frac{1}{3}\Theta h^{tt},$$

$$\sigma^{tr} = \sigma^{rt} = \frac{1}{2}[(u^{t}_{,r} + \Gamma^{t}_{rt}u^{t} + \Gamma^{t}_{r\phi}u^{\phi})h^{rr} + \Gamma^{t}_{tr}u^{r}h^{rt} + (u^{t}_{,\theta} + \Gamma^{t}_{\theta t}u^{t} + \Gamma^{t}_{\theta\phi}u^{\phi})h^{\theta r} + \Gamma^{t}_{\phi r}u^{r}h^{r\phi}$$
$$+(u^{r}_{,r} + \Gamma^{r}_{rr}u^{r})h^{rt} + (\Gamma^{r}_{tt}u^{t} + \Gamma^{r}_{t\phi}u^{\phi})h^{tt} + (u^{r}_{,\theta} + \Gamma^{r}_{r\theta}u^{r})h^{\theta t} + (\Gamma^{r}_{\phi t}u^{t} + \Gamma^{r}_{\phi\phi}u^{\phi})h^{t\phi}] - \frac{1}{3}\Theta h^{rt}$$

$$\sigma^{t\theta} = \sigma^{\theta t} = \frac{1}{2}[(u^{t}_{,r} + \Gamma^{t}_{rt}u^{t} + \Gamma^{t}_{r\phi}u^{\phi})h^{r\theta} + \Gamma^{t}_{tr}u^{r}h^{t\theta} + (u^{t}_{,\theta} + \Gamma^{t}_{\theta t}u^{t} + \Gamma^{t}_{\theta\phi}u^{\phi})h^{\theta\theta} + \Gamma^{t}_{\phi r}u^{r}h^{\theta\phi}$$
$$+(u^{\theta}_{,r} + \Gamma^{\theta}_{rr}u^{r})h^{rt} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta r}u^{r})h^{\theta t} + (\Gamma^{\theta}_{tt}u^{t} + \Gamma^{\theta}_{t\phi}u^{\phi})h^{tt} + (\Gamma^{\theta}_{\phi t}u^{t} + \Gamma^{\theta}_{\phi\phi}u^{\phi})h^{t\phi}] - \frac{1}{3}\Theta h^{t\theta},$$

$$\sigma^{t\phi} = \sigma^{\phi t} = \frac{1}{2}[(u^{t}_{,r} + \Gamma^{t}_{rt}u^{t} + \Gamma^{t}_{r\phi}u^{\phi})h^{r\phi} + \Gamma^{t}_{tr}u^{r}h^{t\phi} + (u^{t}_{,\theta} + \Gamma^{t}_{\theta t}u^{t} + \Gamma^{t}_{\theta\phi}u^{\phi})h^{\theta\phi} + \Gamma^{t}_{\phi r}u^{r}h^{\phi\phi}$$
$$+(u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^{t} + \Gamma^{\phi}_{r\phi}u^{\phi})h^{rt} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^{t} + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta t} + \Gamma^{\phi}_{tr}u^{r}h^{tt} + \Gamma^{\phi}_{\phi r}u^{r}h^{t\phi}] - \frac{1}{3}\Theta h^{t\phi},$$

$$\sigma^{rr} = (u^{r}_{,r} + \Gamma^{r}_{rr}u^{r})h^{rr} + (\Gamma^{r}_{tt}u^{t} + \Gamma^{r}_{t\phi}u^{\phi})h^{rt} + (u^{r}_{,\theta} + \Gamma^{r}_{\theta r}u^{r})h^{\theta r} + (\Gamma^{r}_{\phi t}u^{t} + \Gamma^{r}_{\phi\phi}u^{\phi})h^{r\phi} - \frac{1}{3}\Theta h^{rr},$$

$$\sigma^{r\theta} = \sigma^{\theta r} = \frac{1}{2}[(u^{r}_{,r} + \Gamma^{r}_{rr}u^{r})h^{r\theta} + (\Gamma^{r}_{tt}u^{t} + \Gamma^{r}_{t\phi}u^{\phi})h^{t\theta} + (u^{r}_{,\theta} + \Gamma^{r}_{\theta r}u^{r})h^{\theta\theta} + (\Gamma^{r}_{\phi t}u^{t} + \Gamma^{r}_{\phi\phi}u^{\phi})h^{\theta\phi}$$
$$+(u^{\theta}_{,r} + \Gamma^{\theta}_{rr}u^{r})h^{rr} + (\Gamma^{\theta}_{tt}u^{t} + \Gamma^{\theta}_{t\phi}u^{\phi})h^{rt} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta r}u^{r})h^{\theta r} + (\Gamma^{\theta}_{\phi t}u^{t} + \Gamma^{\theta}_{\phi\phi}u^{\phi})h^{r\phi}] - \frac{1}{3}\Theta h^{r\theta},$$

$$\sigma^{r\phi} = \sigma^{\phi r} = \frac{1}{2}[(u^{r}_{,r} + \Gamma^{r}_{rr}u^{r})h^{r\phi} + (\Gamma^{r}_{tt}u^{t} + \Gamma^{r}_{t\phi}u^{\phi})h^{t\phi} + (u^{r}_{,\theta} + \Gamma^{r}_{\theta r}u^{r})h^{\theta\phi} + (\Gamma^{r}_{\phi t}u^{t} + \Gamma^{r}_{\phi\phi}u^{\phi})h^{\phi\phi}$$
$$+(u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^{t} + \Gamma^{\phi}_{r\phi}u^{\phi})h^{rr} + \Gamma^{\phi}_{tr}u^{r}h^{rt} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^{t} + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta r} + \Gamma^{\phi}_{\phi r}u^{r}h^{r\phi}] - \frac{1}{3}\Theta h^{r\phi},$$

$$\sigma^{\theta\theta} = (u^{\theta}_{,r} + \Gamma^{\theta}_{rr}u^{r})h^{r\theta} + (\Gamma^{\theta}_{tt}u^{t} + \Gamma^{\theta}_{t\phi}u^{\phi})h^{t\theta} + (\Gamma^{\theta}_{\phi t}u^{t} + \Gamma^{\theta}_{\phi\phi}u^{\phi})h^{\theta\phi} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta r}u^{r})h^{\theta\theta} - \frac{1}{3}\Theta h^{\theta\theta},$$

$$\sigma^{\theta\phi} = \sigma^{\phi\theta} = \frac{1}{2}[(u^{\theta}_{,r} + \Gamma^{\theta}_{rr}u^r)h^{r\phi} + (\Gamma^{\theta}_{tt}u^t + \Gamma^{\theta}_{t\phi}u^{\phi})h^{t\phi} + (u^{\theta}_{,\theta} + \Gamma^{\theta}_{\theta r}u^r)h^{\theta\phi} + (\Gamma^{\theta}_{\phi t}u^t + \Gamma^{\theta}_{\phi\phi}u^{\phi})h^{\phi\phi}$$
$$+(u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^t + \Gamma^{\phi}_{r\phi}u^{\phi})h^{\theta r} + \Gamma^{\phi}_{tr}u^r h^{\theta t} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^t + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta\theta} + \Gamma^{\phi}_{\phi r}u^r h^{\theta\phi}] - \frac{1}{3}\Theta h^{\theta\phi},$$
$$\sigma^{\phi\phi} = (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^t + \Gamma^{\phi}_{r\phi}u^{\phi})h^{r\phi} + \Gamma^{\phi}_{tr}u^r h^{t\phi} + (u^{\phi}_{,\theta} + \Gamma^{\phi}_{\theta t}u^t + \Gamma^{\phi}_{\theta\phi}u^{\phi})h^{\theta\phi} + \Gamma^{\phi}_{\phi r}u^r h^{\phi\phi} - \frac{1}{3}\Theta h^{\phi\phi}. \tag{15}$$

5 Bulk and shear tensor in the equatorial plane

In this section, we calculate the components of bulk and shear tensor of relativistic accretion flow in the Kerr metric and in the equatorial plane $\theta = \frac{\pi}{2}$; so, we assume $u^{\theta} = 0$. Then, we have

$$b^{tt} = (u^r_{,r} + \frac{2u^r}{r})(g^{tt} + (u^t)^2), \qquad b^{tr} = b^{rt} = (u^r_{,r} + \frac{2u^r}{r})u^r u^t,$$
$$b^{t\phi} = b^{\phi t} = (u^r_{,r} + \frac{2u^r}{r})(g^{t\phi} + u^t u^{\phi}), \qquad b^{rr} = (u^r_{,r} + \frac{2u^r}{r})(g^{rr} + (u^r)^2),$$
$$b^{r\phi} = b^{\phi r} = (u^r_{,r} + \frac{2u^r}{r})u^r u^{\phi}, \qquad b^{\theta\theta} = (u^r_{,r} + \frac{2u^r}{r})g^{\theta\theta}, \qquad b^{\phi\phi} = (u^r_{,r} + \frac{2u^r}{r})(g^{\phi\phi} + (u^{\phi})^2). \tag{16}$$

$$\sigma^{tt} = (u^t_{,r} + \Gamma^t_{rt}u^t + \Gamma^t_{r\phi}u^{\phi})h^{rt} + \Gamma^t_{tr}u^r u^t u^r + \Gamma^t_{\phi r}u^r h^{t\phi} - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})h^{tt},$$
$$\sigma^{tr} = \sigma^{rt} = \frac{1}{2}[(u^t_{,r} + \Gamma^t_{rt}u^t + \Gamma^t_{r\phi}u^{\phi})h^{rr} + \Gamma^t_{tr}u^r u^t u^r + \Gamma^t_{\phi r}u^r u^r u^{\phi} + (u^r_{,r} + \Gamma^r_{rr}u^r)u^t u^r$$
$$+(\Gamma^r_{tt}u^t + \Gamma^r_{t\phi}u^{\phi})h^{tt} + (\Gamma^r_{\phi t}u^t + \Gamma^r_{\phi\phi}u^{\phi})h^{t\phi}] - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})u^t u^r,$$
$$\sigma^{t\phi} = \sigma^{\phi t} = \frac{1}{2}[(u^t_{,r} + \Gamma^t_{rt}u^t + \Gamma^t_{r\phi}u^{\phi})u^r u^{\phi} + \Gamma^t_{tr}u^r h^{t\phi} + \Gamma^t_{\phi r}u^r h^{\phi\phi} + (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^t + \Gamma^{\phi}_{r\phi}u^{\phi})u^t u^r$$
$$+\Gamma^{\phi}_{tr}u^r h^{tt} + \Gamma^{\phi}_{\phi r}u^r h^{t\phi}] - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})h^{t\phi},$$
$$\sigma^{rr} = (u^r_{,r} + \Gamma^r_{rr}u^r)h^{rr} + (\Gamma^r_{tt}u^t + \Gamma^r_{t\phi}u^{\phi})u^t u^r + (\Gamma^r_{\phi t}u^t + \Gamma^r_{\phi\phi}u^{\phi})u^r u^{\phi} - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})h^{rr},$$
$$\sigma^{r\phi} = \sigma^{\phi r} = \frac{1}{2}[(u^r_{,r} + \Gamma^r_{rr}u^r)u^r u^{\phi} + (\Gamma^r_{tt}u^t + \Gamma^r_{t\phi}u^{\phi})h^{t\phi} + (\Gamma^r_{\phi t}u^t + \Gamma^r_{\phi\phi}u^{\phi})h^{\phi\phi}$$
$$+(u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^t + \Gamma^{\phi}_{r\phi}u^{\phi})h^{rr} + \Gamma^{\phi}_{tr}u^r u^t u^r + \Gamma^{\phi}_{\phi r}u^r u^r u^{\phi}] - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})u^r u^{\phi},$$
$$\sigma^{\theta\theta} = -\frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})g^{\theta\theta},$$
$$\sigma^{\phi\phi} = (u^{\phi}_{,r} + \Gamma^{\phi}_{rt}u^t + \Gamma^{\phi}_{r\phi}u^{\phi})h^{r\phi} + \Gamma^{\phi}_{tr}u^r u^r u^{\phi} + \Gamma^{\phi}_{\phi r}u^r h^{\phi\phi} - \frac{1}{3}(u^r_{,r} + \frac{2u^r}{r})h^{\phi\phi}. \tag{17}$$

6 Sample radial model for radial component of four-velocity

To see the behavior of the bulk tensor and shear tensor of the relativistic accretion disks around the rotating black holes, we want to use a suitable model for the four-velocity. At present, to the best of our knowledge, there is no suggested model for four-velocity.

In the self-similar solution of non-relativistic accretion disks, $v_r = v_0 r^{-n}, n \geq 0$ are used for the radial velocity ,where v_0 and n are constant which are derived from the basic equation of accretion disk and in the most paper, $n = \frac{1}{2}$ are used. In the non-relativistic fluids, the space time metric is flat and proper time is equal to time, so

$$u^t = \frac{dt}{d\tau} = 1, v_r = \frac{u^r}{u^t} \Rightarrow u^r = v_0 r^{-n}. \tag{18}$$

In the some previous papers of the relativistic accretion disks, we see that the radial components of the four-velocity has the decreasing function of radius [5, 12]. Also, locally non-rotating frame (LNRF) has the flat metric and in LNRF $u^t \geq 1$. Therefore, we believe that the decreasing function of radius for the radial components of the four-velocity in LNRF will be a suitable model. So, we assume a radial model for the radial components of the four-velocity in the LNRF as

$$u^{\hat{r}} = -\frac{\beta}{r^n}, \tag{19}$$

β and n are positive and constant and minus sign is for the direction of $u^{\hat{r}}$ which is toward of the black hole. With the transformations, we can calculate the radial component of the four-velocity in the BLF; so, this component in BLF is given as

$$u^r = e^{\nu}_{\hat{\mu}} \delta^r_\nu u^{\hat{\mu}}, \tag{20}$$

where $e^{\nu}_{\hat{\mu}}$ are the components of connecting between the BLF and the LNRF which are given in the Appendix A. In the Appendix A, we see that $e^r_{\hat{t}} = e^r_{\hat{\theta}} = e^r_{\hat{\phi}} = 0$; so, the radial velocity in the BLF will be as

$$u^r = e^r_{\hat{\mu}} u^{\hat{\mu}} = e^r_{\hat{r}} u^{\hat{r}} = \sqrt{\frac{\Delta}{\Sigma}} u^{\hat{r}} = -\frac{\beta\sqrt{r^2 - 2r + a^2}}{r^{n+1}}. \tag{21}$$

The Keplerian angular momentum are given as

$$\Omega^{\pm}_k = \pm \frac{1}{r^{\frac{3}{2}} \pm a}, \tag{22}$$

where $(+)$ are used to the angular frequencies of the corotating Keplerian orbits and $(-)$ are used to the angular frequencies of the counterrotating Keplerian orbits [1]. If we assume the corotating Keplerian angular momentum, we have

$$\Omega = \frac{u^\phi}{u^t} = \Omega^+_k = \frac{1}{r^{\frac{3}{2}} + a}. \tag{23}$$

Using the normalization condition for four-velocity, that is $u^\mu u_\mu = -1$, we have

$$\begin{aligned} &-1 = u^t u_t + u^r u_r + u^\phi u_\phi \\ &\Rightarrow -1 = u^t(g_{tt}u^t + g_{t\phi}u^\phi) + g_{rr}(u^r)^2 + u^\phi(g_{\phi\phi}u^\phi + g_{t\phi}u^t) \\ &\Rightarrow -1 = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\phi\phi}(\Omega)^2(u^t)^2 + 2g_{t\phi}\Omega(u^t)^2. \end{aligned} \tag{24}$$

By using equations (21) and (23) and substituting them into equation (24), we obtain

$$u^t = \frac{r^{\frac{3}{2}} + a}{r^n}\sqrt{\frac{r^{2n+1} + r\beta^2}{r^4 + 2ar^{\frac{5}{2}} - 3r^3}}. \tag{25}$$

6.1 Checking velocity with time dilation

The time dilation is a relativistic influence which shows the coordinate time (t) is larger than the proper time (τ). The time dilation is derived by putting $u^r = u^\theta = u^\phi = 0$. This parameter is minimum value of u^t, so in each assumed model u^t must be checked, that is, $u^t \geq u^{t0}$, where from Moeen[8] we have

$$u^{t0} = u^t_{min} = \sqrt{|1/g_{tt}|}. \tag{26}$$

Therefore,

$$\begin{aligned}
&\frac{r^{\frac{3}{2}}+a}{r^n}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^4+2ar^{\frac{5}{2}}-3r^3}} \geq \sqrt{\frac{r}{r-2}} \Rightarrow \frac{(r^{\frac{3}{2}}+a)^2(r^{2n+1}+r\beta^2)}{r^{2n}(r^4+2ar^{\frac{5}{2}}-3r^3)} \geq \frac{r}{r-2} \\
&\Rightarrow r^{2n+4} - 4ar^{\frac{3}{2}}r^{2n+1} + r^5\beta^2 - 2r^4\beta^2 + 2ar^{\frac{3}{2}}r^2\beta^2 - 4ar^{\frac{5}{2}}\beta^2 + r^2a^2\beta^2 - 2ra^2\beta^2 \\
&+a^2r^{2n+2} - 2a^2r^{2n+1} \geq 0.
\end{aligned} \tag{27}$$

The ergosphere is a region where everything are dragged into the black hole or forced to rotate with black hole; so, the accretion disks are studied outside the ergosphere. In the equatorial plan in the BLF, the ergosphere lies at $r = 2$ [5]; so, we exam the relation (23) in $r \geq 2$.

In this relation, all terms are positive except the last three terms which are negative. But in $r \geq 2$, the expressions $r^{2n+4} \geq 4ar^{\frac{3}{2}}r^{2n+1}$, $r^5\beta^2 \geq 2r^4\beta^2$, $2ar^{\frac{3}{2}}r^2\beta^2 \geq 4ar^{\frac{5}{2}}\beta^2$, $r^2a^2\beta^2 \geq 2ra^2\beta^2$ and $a^2r^{2n+2} \geq 2a^2r^{2n+1}$ are being true; therefore, outside the ergosphere the expression $u^t \geq u^{t0}$ is confirmed; so, the four-velocity in the BLF can be written as follows

$$u^\mu = (\frac{r^{\frac{3}{2}}+a}{r^n}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^4+2ar^{\frac{5}{2}}-3r^3}}, -\frac{\beta\sqrt{r^2-2r+a^2}}{r^{n+1}}, 0, \frac{1}{r^n}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^4+2ar^{\frac{5}{2}}-3r^3}}). \tag{28}$$

7 Shear and bulk tensors

In this section, the radial model is used to calculate the components of four velocity in the BLF. Then, the components of the shear and the bulk tensors for some sample values of parameter n are derived.

I) Solutions for $n = \frac{1}{2}$:

The four-velocity and the expansion of the fluid world line for $n = \frac{1}{2}$ are derived with equations (28) and (8) as

$$\begin{aligned}
u^\mu &= (\frac{\sqrt{r^2+r\beta^2}(r^{\frac{3}{2}}+a)}{\sqrt{r}\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}}, -\frac{\beta\sqrt{r^2-2r+a^2}}{r^{\frac{3}{2}}}, 0, \frac{\sqrt{r^2+r\beta^2}}{\sqrt{r}\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}}), \\
\Theta &= -\frac{\beta(-4r+3r^2+a^2)}{2r^{\frac{5}{2}}\sqrt{r^2-2r+a^2}}.
\end{aligned} \tag{29}$$

Also, the components of the shear and the bulk tensors versus dimensionless parameter r/m are shown in figure 2.

II) Solutions for $n = 1$:

For $n=1$, the four-velocity and the expansion of the fluid world line are derived as

$$u^{\mu}=(\frac{\sqrt{r^3+r\beta^2}(r^{\frac{3}{2}}+a)}{r\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}},-\frac{\beta\sqrt{r^2-2r+a^2}}{r^2},0,\frac{\sqrt{r^3+r\beta^2}}{r\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}}),$$
$$\Theta=-\frac{\beta(r-1)}{r^2\sqrt{r^2-2r+a^2}}. \quad (30)$$

In the figure 3, the components of the shear and the bulk tensors versus dimensionless parameter r/m are shown.

III) Solutions for $n=2$:

The four-velocity and the expansion of the fluid world line for $n=2$ are obtained as follows

$$u^{\mu}=(\frac{\sqrt{r^5+r\beta^2}(r^{\frac{3}{2}}+a)}{r^2\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}},-\frac{\beta\sqrt{r^2-2r+a^2}}{r^3},0,\frac{\sqrt{r^5+r\beta^2}}{r^2\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}}),$$
$$\Theta=\frac{\beta(-r+a^2)}{r^4\sqrt{r^2-2r+a^2}}. \quad (31)$$

Figure 4 shows the components of shear and bulk tensors versus dimensionless parameter r/m for $n=2$.

IV) Solutions for $n=3$

In this state, the four-velocity and the expansion of the fluid world line are calculated as

$$u^{\mu}=(\frac{\sqrt{r^7+r\beta^2}(r^{\frac{3}{2}}+a)}{r^3\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}},-\frac{\beta\sqrt{r^2-2r+a^2}}{r^4},0,\frac{\sqrt{r^7+r\beta^2}}{r^3\sqrt{r^4+2ar^{\frac{5}{2}}-3r^3}}),$$
$$\Theta=\frac{\beta(-3r+r^2+2a^2)}{r^5\sqrt{r^2-2r+a^2}}. \quad (32)$$

All the components of the shear and bulk tensors versus dimensionless parameter r/m are seen in figure 5.

Figure 1 shows the expansion of fluid world line (Θ) for $n=\frac{1}{2},1,2,3$ and for $a=0.9,0.5,0.1,0$. It is shown that increasing the value of the parameter n and decreasing the value of the parameter a, decreases the expansion of fluid word line.

For $n=\frac{1}{2}$ and $a=0.9,0.5,0.1,0$, except $r\phi$ component, the amounts of the other components of the bulk tensor are greater than or about the amounts of components of the shear tensor. By increasing n, the bulk tensor and also the shear tensor are been smaller. So, we see that for $n=1$ and for $a=0.9,0.5,0.1,0$ the amount of tt, $t\phi$, $\phi\phi$, $\theta\theta$ components of the bulk tensor are greater or about of the shear tensor. Also, for the bigger $n(n=2,3)$ and for $a=0.9,0.5,0.1,0$ the amounts of components of the shear tensor and bulk tensor are near zero and the shear tensor is greater than the bulk tensor in the most components.

8 Summary and conclusions

In this paper, we try to calculate the relativistic shear and bulk tensor. The relations of all the components of the relativistic bulk tensor and the shear tensor in the Kerr metric without any approximation are derived. The relativistic bulk tensor, $b^{\mu\nu}$ is created by radial

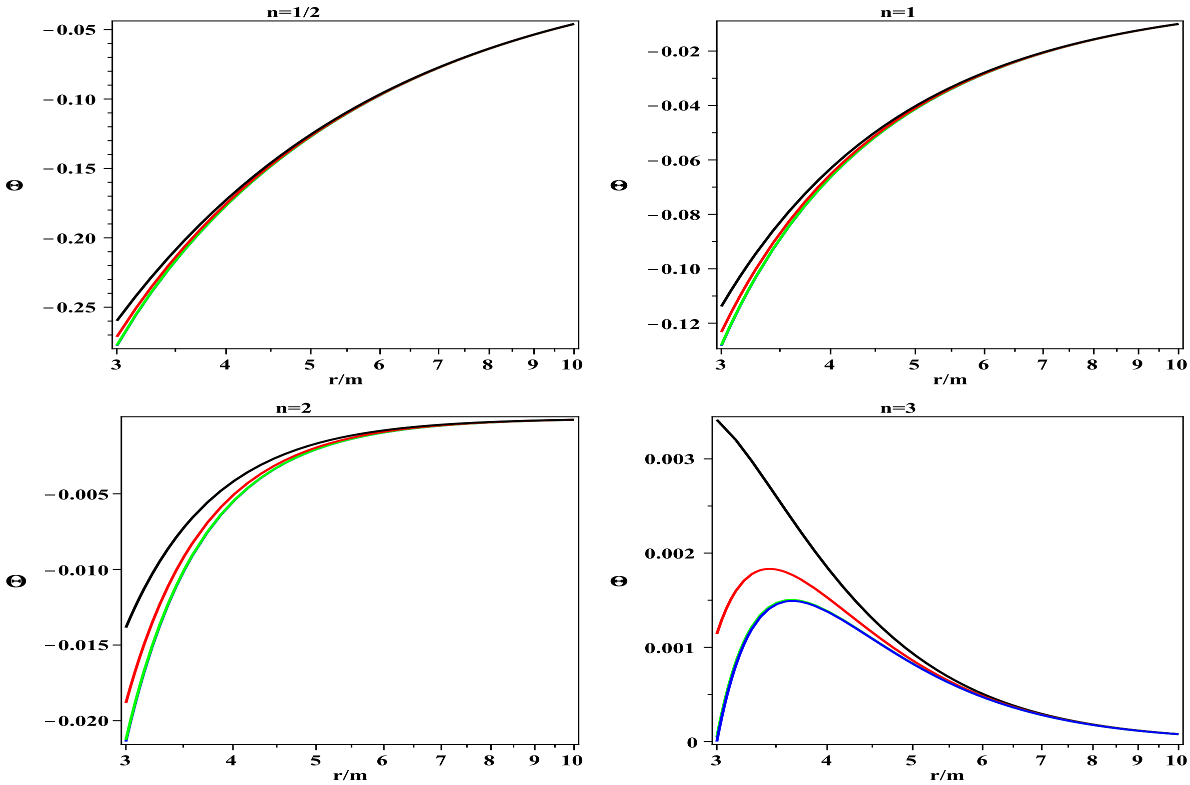

Figure 1: The expansion of fluid world line tensor versus dimensionless parameter r/m for $\beta = 1$ and $n = \frac{1}{2}, 1, 2$ and 3. The black, red, green and blue curves are corresponding to the cases with $a = .9, a = .5, a = .1$ and $a = 0$ respectively.

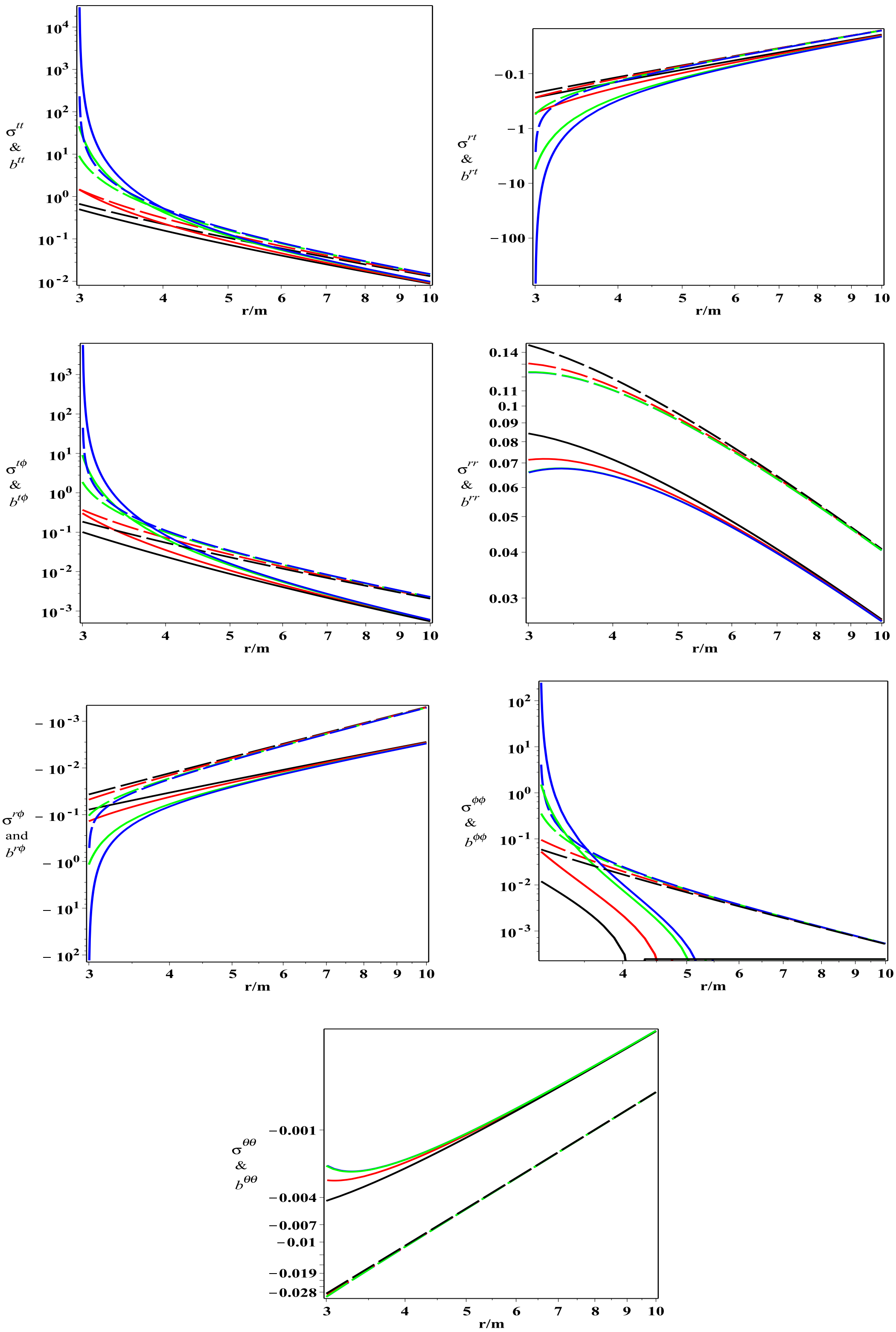

Figure 2: The non-zero components of the shear(Solid curves) and the bulk(dotted curves) tensors versus dimensionless parameter r/m with $\beta = 1$ and $n = \frac{1}{2}$.The black, red, green and blue curves are corresponding to the cases with $a = .9, a = .5, a = .1$ and $a = 0$ respectively.

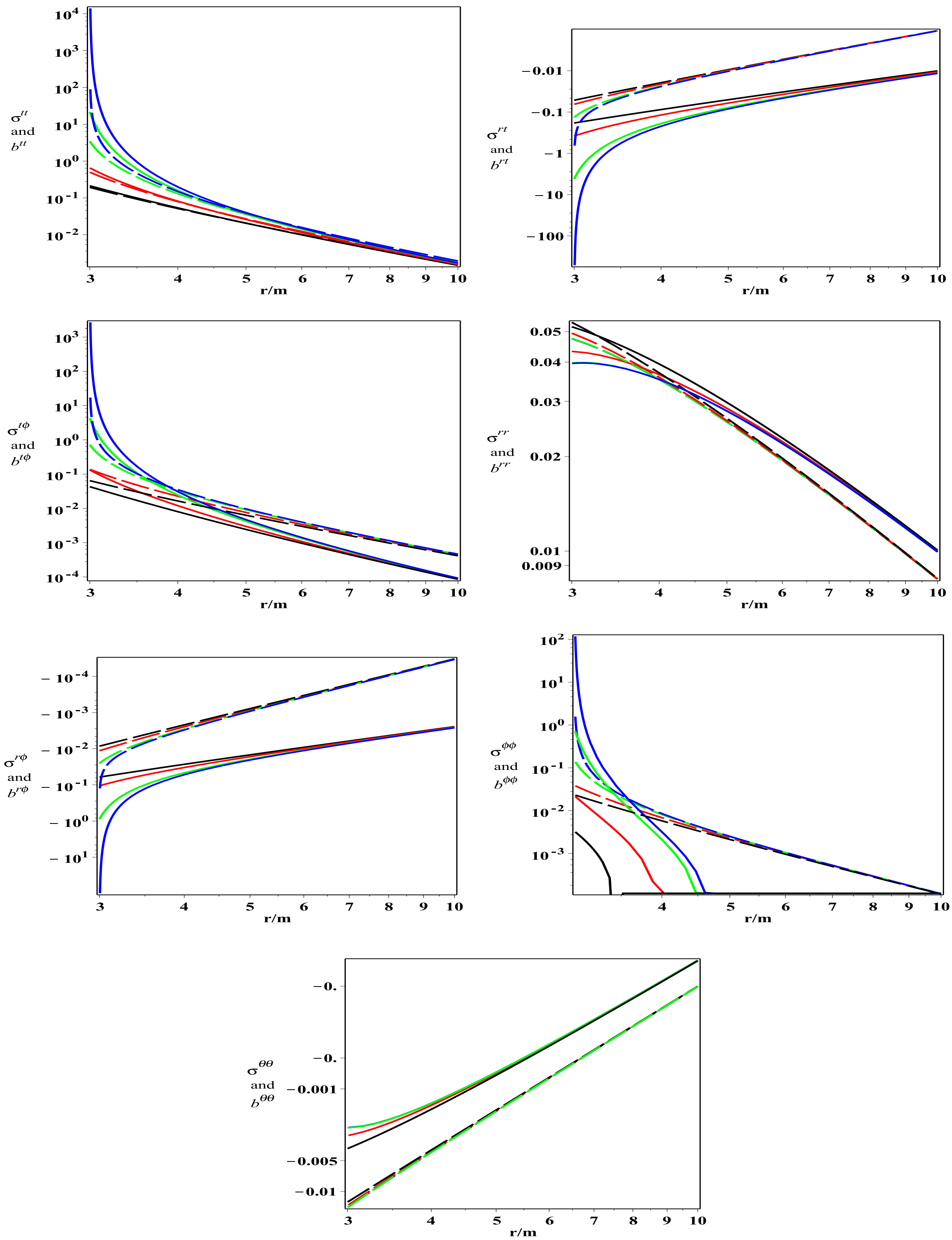

Figure 3: The non-zero components of the shear(Solid curves) and the bulk(dotted curves) tensors versus dimensionless parameter r/m with $\beta = 1$ and $n = 1$. The black, red, green and blue curves are corresponding to the cases with $a = .9, a = .5, a = .1$ and $a = 0$ respectively.

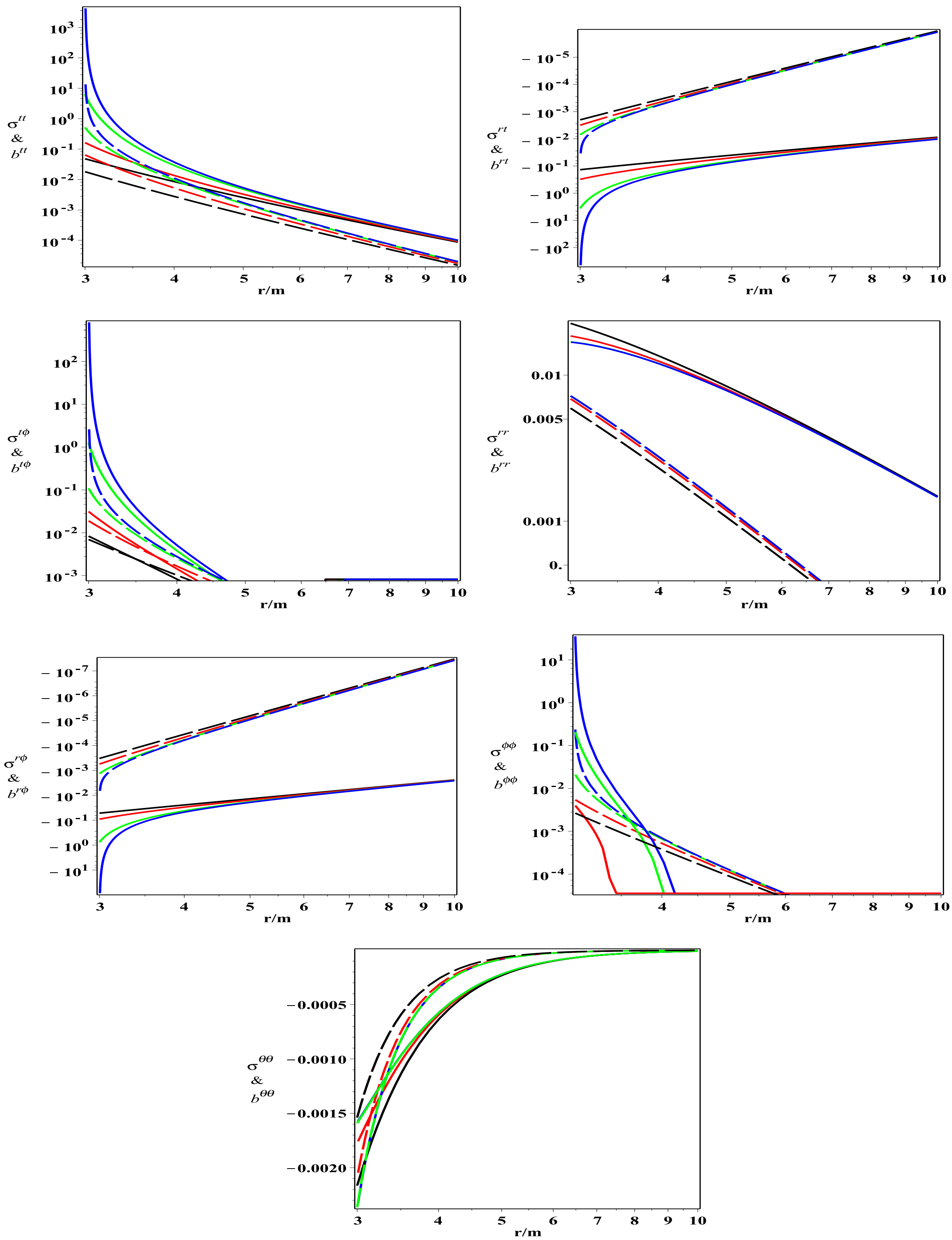

Figure 4: The non-zero components of the shear(Solid curves) and the bulk(dotted curves) tensors versus dimensionless parameter r/m with $\beta = 1$ and $n = 2$. The black, red, green and blue curves are corresponding to the cases with $a = .9, a = .5, a = .1$ and $a = 0$ respectively.

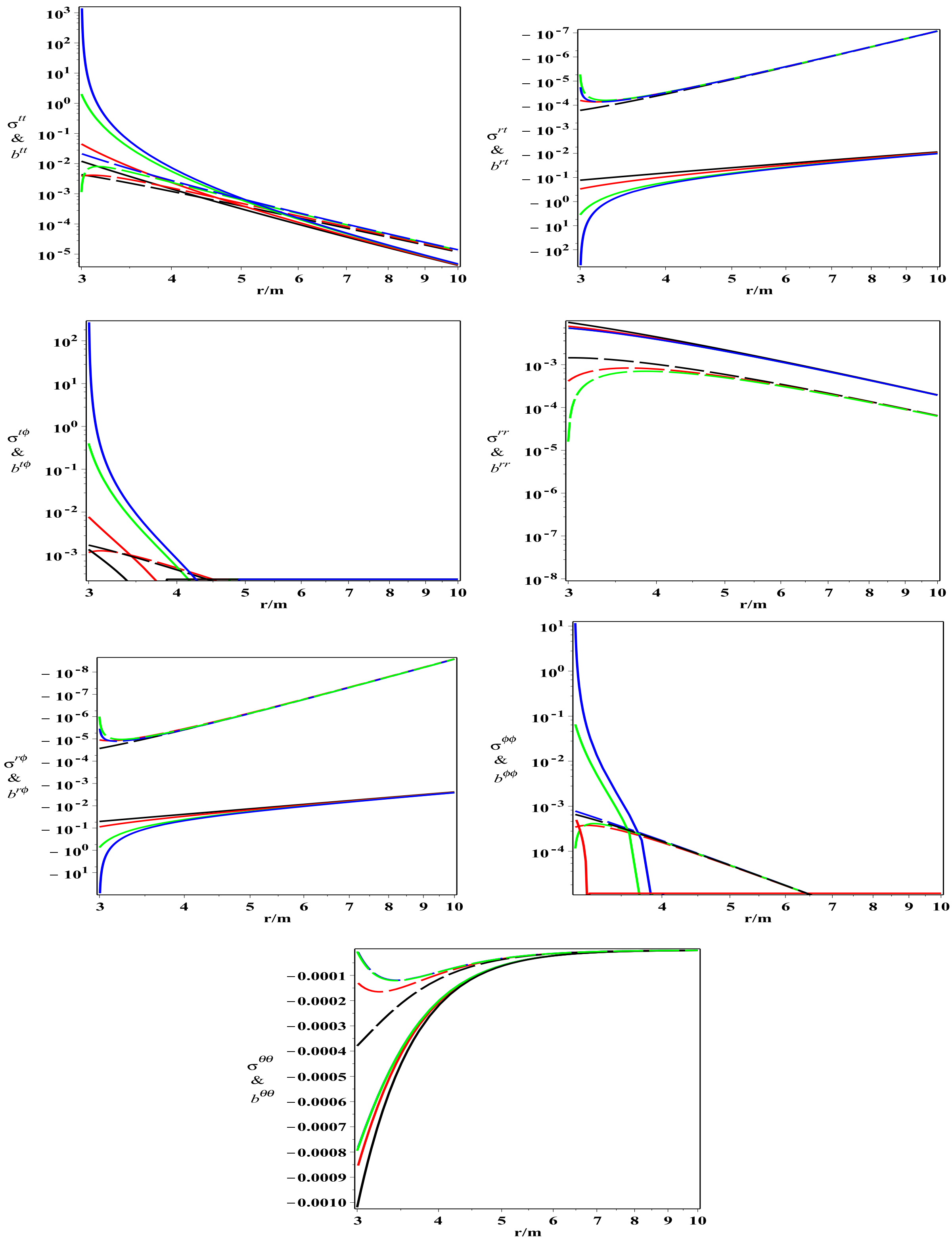

Figure 5: The non-zero components of the shear(Solid curves) and the bulk(dotted curves) tensors versus dimensionless parameter r/m with $\beta = 1$ and $n = 3$. The black, red, green and blue curves are corresponding to the cases with $a = .9, a = .5, a = .1$ and $a = 0$ respectively.

variation of the radial velocity and components of u^μ and u^ν. But the relativistic shear tensor, $\sigma^{\mu\nu}$ is created by radial variation of u^μ, u^ν, u^r and all the components of the four-velocity. Therefore, we expect in the most cases the relativistic shear tensor is larger than the relativistic bulk tensor except in the big radial four-velocity.

In the equatorial plane, there are ten non-zero components for shear and bulk tensor. Ten non-zero components of the relativistic bulk tensor and shear tensor are tt, tr, $t\phi$, rt, rr, $r\phi$, $\theta\theta$, ϕt, ϕr and $\phi\phi$ components. We introduce the sample radial model for the radial four-velocity in the LNRF and with transformation, we calculate the components of the four-velocity in the BLF in four cases. With the relativistic calculation, all the non-zero components of the bulk and shear tensor versus dimensionless parameter r/m are derived. These components are shown in figures 2-5. In these figures, we find that in the smaller n and bigger a, the bulk tensor may be comparable with the shear tensor. Because in those cases, the value of the radial four-velocity and expansion of the fluid world line are bigger.

We use the radial model for the radial velocity. In this model comparison of figures shows that the bulk tensor is important in bigger a(spin of black hole $a = Jc/GM^2$), lower n and in the inner radii, therefore if the radial four-velocity has the similar form of this model, the bulk viscosity will be important. So, in all cases, the bulk tensor is not ignorable.

We know that the expansion of the fluid world line is equal to the divergence of velocity. Also, in the fluid dynamics, the continuity equation ($\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$), shows that in the incompressible flow(with the constant density), $\nabla \cdot u = 0$. So, the non-zero bulk tensor i.e., $\Theta = \nabla \cdot u \neq 0$, shows that the flow may be compressible because the bulk viscosity due to volume movement of fluid. Figures show that the bulk tensor is more important in the smaller n and inner radii. Therefore, in bigger n and outer radii we can suppose the incompressible fluid.

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A Transformation Between Boyer-Lindquist frame and locally non-rotating frame

We can transfer the physical quantities between BLF and the LNRF by using the connecting tensors ($e_{\nu}^{\hat{\mu}}$ and $e_{\hat{\nu}}^{\mu}$). The components of connecting tensors are given as [2]; [3]; [4]

$$\begin{pmatrix} e_t^{\hat{t}} & e_t^{\hat{r}} & e_t^{\hat{\theta}} & e_t^{\hat{\phi}} \\ e_r^{\hat{t}} & e_r^{\hat{r}} & e_r^{\hat{\theta}} & e_r^{\hat{\phi}} \\ e_\theta^{\hat{t}} & e_\theta^{\hat{r}} & e_\theta^{\hat{\theta}} & e_\theta^{\hat{\phi}} \\ e_\phi^{\hat{t}} & e_\phi^{\hat{r}} & e_\phi^{\hat{\theta}} & e_\phi^{\hat{\phi}} \end{pmatrix} = \begin{pmatrix} (\frac{\Sigma\Delta}{A})^{\frac{1}{2}} & 0 & 0 & -\frac{2Mar\sin\theta}{(\Sigma A)^{\frac{1}{2}}} \\ 0 & (\frac{\Sigma}{\Delta})^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \Sigma^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & (\frac{A}{\Sigma})^{\frac{1}{2}}\sin\theta \end{pmatrix}, \tag{33}$$

$$\begin{pmatrix} e_{\hat{t}}^{t} & e_{\hat{r}}^{t} & e_{\hat{\theta}}^{t} & e_{\hat{\phi}}^{t} \\ e_{\hat{t}}^{r} & e_{\hat{r}}^{r} & e_{\hat{\theta}}^{r} & e_{\hat{\phi}}^{r} \\ e_{\hat{t}}^{\theta} & e_{\hat{r}}^{\theta} & e_{\hat{\theta}}^{\theta} & e_{\hat{\phi}}^{\theta} \\ e_{\hat{t}}^{\phi} & e_{\hat{r}}^{\phi} & e_{\hat{\theta}}^{\phi} & e_{\hat{\phi}}^{\phi} \end{pmatrix} = \begin{pmatrix} (\frac{A}{\Sigma\Delta})^{\frac{1}{2}} & 0 & 0 & \frac{2Mar}{(\Sigma A\Delta)^{\frac{1}{2}}} \\ 0 & (\frac{\Delta}{\Sigma})^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & (\frac{\Sigma}{A})^{\frac{1}{2}}\frac{1}{\sin\theta} \end{pmatrix}, \tag{34}$$

where hat $(\hat{})$ is used for the quantities in the LNRF.

B The Christoffel symbols in the BLF

After some calculation the Christoffel symbols ($\Gamma^{\alpha}_{\beta\gamma}$) in the equatorial plane and in our scaling of the BLF are

$$\Gamma^t_{tt}=0,\quad \Gamma^t_{t\phi}=\Gamma^t_{\phi t}=0,\quad \Gamma^t_{rr}=0,\quad \Gamma^t_{r\theta}=\Gamma^t_{\theta r}=0,\quad \Gamma^t_{\theta\theta}=0,\quad \Gamma^t_{\phi\phi}=0$$

$$\Gamma^r_{tr}=\Gamma^r_{rt}=0,\quad \Gamma^r_{t\theta}=\Gamma^r_{\theta t}=0,\quad \Gamma^r_{r\phi}=\Gamma^r_{\phi r}=0,\quad \Gamma^r_{\theta\phi}=\Gamma^r_{\phi\theta}=0,\quad \Gamma^\theta_{tr}=\Gamma^\theta_{rt}=0,$$

$$\Gamma^\theta_{t\theta}=\Gamma^\theta_{\theta t}=0,\quad \Gamma^\theta_{r\phi}=\Gamma^\theta_{\phi r}=0,\quad \Gamma^\theta_{\theta\phi}=\Gamma^\theta_{\phi\theta}=0,\quad \Gamma^\phi_{tt}=0,\quad \Gamma^\phi_{rr}=0,$$

$$\Gamma^\phi_{r\theta}=\Gamma^\phi_{\theta r}=0,\quad \Gamma^\phi_{\theta\theta}=0,\quad \Gamma^\phi_{t\phi}=\Gamma^\phi_{\phi t}=0,\quad \Gamma^\phi_{\phi\phi}=0,$$

$$\Gamma^t_{rt}=\Gamma^t_{tr}=\frac{(-\Sigma+r\Sigma_{,r})A-2a^2r+2a^2rcos^2\theta}{\Sigma^3\Delta}\qquad \Gamma^t_{r\phi}=\Gamma^t_{\phi r}=-\frac{a(a^2+3r^2)}{r^2(a^2+r^2-2r)},$$

$$\Gamma^t_{t\theta}=\Gamma^t_{\theta t}=\frac{r(A\Sigma_{,r}+4a^2rsin\theta cos\theta\Sigma-2a^2r\Sigma_{,\theta}+2a^2r\Sigma_{,\theta}cos^2\theta)}{\Sigma^3\Delta}\qquad \Gamma^r_{tt}=\frac{a^2+r^2-2r}{r^4},$$

$$\Gamma^t_{\theta\phi}=\Gamma^t_{\phi\theta}=-\frac{arsin^2\theta A_{,\theta}}{\Sigma^2\Delta},\qquad \Gamma^r_{t\phi}=\Gamma^r_{\phi t}=\frac{\Delta asin^2\theta(\Sigma-r\Sigma_{,r})}{\Sigma^3},\qquad \Gamma^r_{rr}=-\frac{-\Sigma_{,r}\Delta+\Sigma\Delta_{,r}}{2\Delta\Sigma}$$

$$\Gamma^r_{r\theta}=\Gamma^r_{\theta r}=\frac{\Sigma_{,\theta}}{2\Sigma},\qquad \Gamma^r_{\theta\theta}=-\frac{\Delta\Sigma_{,r}}{2\Sigma},\qquad \Gamma^r_{\theta\theta}=-\frac{\Delta sin^2\theta(A_{,r}\Sigma-A\Sigma_{,r})}{2\Sigma^3},\qquad \Gamma^\theta_{tt}=\frac{r\Sigma_{,\theta}}{\Sigma^3},$$

$$\Gamma^\theta_{t\phi}=\Gamma^\theta_{\phi t}=\frac{arsin\theta(2cos\theta\Sigma-sin\theta\Sigma_{,\theta})}{\Sigma^3},\qquad \Gamma^\theta_{rr}=-\frac{\Sigma_{,\theta}}{2\Sigma\Delta},\qquad \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\frac{\Sigma_{,r}}{2\Sigma},$$

$$\Gamma^\theta_{\theta\theta}=\frac{\Sigma_{,\theta}}{2\Sigma},\qquad \Gamma^\theta_{\phi\phi}=-\frac{sin\theta(A_{,\theta}sin\theta\Sigma+2Acos\theta\Sigma-Asin\theta\Sigma_{,\theta})}{2\Sigma^3},$$

$$\Gamma^\phi_{tr}=\Gamma^\phi_{rt}=\frac{a(-\Sigma+r\Sigma_{,r})}{\Sigma^2\Delta},\qquad \Gamma^\phi_{t\theta}=\Gamma^\phi_{\theta t}=-\frac{ar(2cos\theta\Sigma-sin\theta\Sigma_{,\theta})-4rcos\theta)}{\Sigma^2\Delta sin\theta},$$

$$\Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{2\Sigma^3\Delta}(4a^2r\Sigma-4a^2r\Sigma cos^2\theta-4a^2r^2\Sigma_{,r}+4a^2r^2\Sigma_{,r}cos^2\theta+A_{,r}\Sigma^2-\Sigma A\Sigma_{,r}$$

$$-2rA_{,r}\Sigma+2rA\Sigma_{,r}),\qquad \Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=-\frac{1}{2\Delta sin\theta\Sigma^3}(-8a^2r^2cos\theta\Sigma+8a^2r^2cos^3\theta\Sigma$$

$$+4a^2r^2 sin\theta\Sigma_{,\theta} - 4a^2r^2 sin\theta\Sigma_{,\theta} cos^2\theta - A_{,\theta} sin\theta\Sigma^2 - 2Acos\theta\Sigma^2 + \Sigma A sin\theta\Sigma_{,\theta} + 2rA_{,\theta} sin\theta\Sigma$$
$$+4rAcos\theta\Sigma - 2rAsin\theta\Sigma_{,\theta}). \quad (35)$$